

# ON ARRANGEMENTS OF PSEUDOHYPERPLANES

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**ABSTRACT.** We study arrangements of pseudohyperplanes (hyperplanes that are topologically deformed in some mild way). In general these arrangements correspond to non-realizable oriented matroids and arise as a consequence of the Folkman-Lawrence Topological Representation Theorem. We introduce a topological space naturally associated with these pseudo arrangements that has the homotopy type of the associated Salvetti complex.

## INTRODUCTION

An arrangement of hyperplanes is a finite set  $\mathcal{A}$  consisting of linear codimension 1 subspaces of  $\mathbb{R}^l$ . These hyperplanes and their intersections induce a stratification of  $\mathbb{R}^l$ . The strata (or faces) form a poset (face poset) when ordered by inclusion and the set of all possible intersections forms a poset ordered by reverse inclusion. These posets contain important combinatorial information about the arrangement. An important topological object associated with an arrangement  $\mathcal{A}$  is the *complexified complement*  $M(\mathcal{A})$ . It is the complement of the union of the complexified hyperplanes in  $\mathbb{C}^l$ . One of the important aspects of the theory of arrangements is to understand the interaction between the combinatorial data of an arrangement and the topology of this complement. For example, a pioneering result by Salvetti [22] states that the homotopy type of the complement is determined by the face poset. He constructed a regular cell complex (now known as the Salvetti complex), using the incidence relations of faces, on which the complement deformation retracts.

On the other hand the oriented matroids are intimately connected with hyperplane arrangements. Oriented matroids not only provide a combinatorial structure that combines the above mentioned posets but they also supply rich techniques to study arrangements. The strata of a hyperplane arrangement satisfy the covector axioms of an oriented matroid. The oriented matroids which correspond to faces of a hyperplane arrangement are known as the *realizable oriented matroids*. There are oriented matroids that do not correspond to hyperplane arrangements (e.g. non-Pappus configuration). Hence for a long time an important question in this field was to come up with the right topological model for oriented matroids. This was settled by Folkman and Lawrence in [12]. The *Folkman-Lawrence Topological Representation Theorem* states that oriented matroids are completely realizable in terms of geometric topology: they may not correspond to real hyperplane arrangements, but they correspond to certain collections of topological spheres and balls (i.e. *arrangements of pseudo-hemispheres*). These pseudo arrangements not only create oriented matroids in the same way that  $\mathbb{R}^l$  and collections of half spaces create an obvious combinatorial structure but there is a one-to-one correspondence between such arrangements and the oriented matroids. In his thesis Mandel [16] introduced

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“sphere systems” that simplified some aspects of the pseudo-hemisphere arrangements and also proved the stronger piecewise linear version of the representation theorem.

In his thesis Ziegler [24, Section 5.5] extended the definition of the Salvetti complex to arbitrary oriented matroids. To every oriented matroid one can associate a simplicial complex and in case of a realizable oriented matroid this complex has the homotopy type of the space  $M(\mathcal{A})$ . In their paper Gel’fand and Rybnikov [13] studied the Salvetti complex for arbitrary oriented matroids and showed that the cohomology ring of this complex is isomorphic to the Orlik-Solomon algebra of the associated lattice of flats (see also [3]). This result not only extends the classical theorem of Brieskorn and Orlik-Solomon but also gives a completely combinatorial proof.

An important thing missing in this study is a twice-dimensional space naturally associated with the pseudo arrangements that has the homotopy type of the associated Salvetti complex, i.e. a generalization of the complexified complement. The aim of this paper is to introduce such a space.

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## 1. ARRANGEMENTS AND ORIENTED MATROIDS

**1.1. Basics of Hyperplane Arrangements.** Hyperplane arrangements arise naturally in geometric, algebraic and combinatorial instances. They occur in various settings such as finite dimensional projective or affine (vector) spaces defined over field of any characteristic. In this section we will formally define hyperplane arrangements and the combinatorial data associated with it in a setting that is most relevant to our work.

**Definition 1.1.** A real, central *arrangement of hyperplanes* is a collection  $\mathcal{A} = \{H_1, \dots, H_k\}$  of finitely many codimension 1 subspaces (*hyperplanes*) in  $\mathbb{R}^l$ ,  $l \geq 1$ . Here  $l$  is called as the *rank* of the arrangement.

If we allow  $\mathcal{A}$  to contain affine hyperplanes (i.e., translates of codimension 1 subspaces) we call  $\mathcal{A}$  an affine arrangement. However we will mostly consider central arrangements. Hence, an arrangement will always mean central, unless otherwise stated. We also assume that all our arrangements are *essential*, it means that the intersection of all the hyperplanes is the origin. For an affine subspace  $X$  of  $\mathbb{R}^l$ , the *contraction* of  $X$  in  $\mathcal{A}$  is given by the sub-arrangement  $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$ . The hyperplanes of  $\mathcal{A}$  induce a stratification (cellular decomposition) on  $\mathbb{R}^l$ , components of each stratum are called *faces*.

There are two posets associated with  $\mathcal{A}$ , namely, the face poset and the intersection lattice which contain important combinatorial information about the arrangement.

**Definition 1.2.** The *intersection lattice*  $L(\mathcal{A})$  of  $\mathcal{A}$  is defined as the set of all intersections of hyperplanes ordered by reverse inclusion.

$$L(\mathcal{A}) := \{X := \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}, X \neq \emptyset\}, \quad X \geq Y \Leftrightarrow X \subseteq Y$$

Note that for affine arrangements, set of all intersections only form a poset and not a lattice.

**Definition 1.3.** The *face poset*  $\mathcal{F}(\mathcal{A})$  of  $\mathcal{A}$  is the set of all faces ordered by inclusion:  $F \leq G$  if and only if  $F \subseteq \overline{G}$ .

Codimension 0 faces are called *chambers*, the set of all chambers will be denoted by  $\mathcal{C}(\mathcal{A})$ . A chamber is bounded if and only if it is a bounded subset of  $\mathbb{R}^l$ . Two chambers  $C$  and  $D$  are adjacent if they have a common face.

**Definition 1.4.** Let  $\mathcal{A}$  denote a real hyperplane arrangement its *complexified complement*  $M(\mathcal{A})$  is defined as follows:

$$M(\mathcal{A}) := \mathbb{C}^l \setminus \left( \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right)$$

where  $H_{\mathbb{C}}$  is the hyperplane in  $\mathbb{C}^l$  with the same defining equation as  $H \in \mathcal{A}$ .

There is a combinatorial description of the homotopy type of  $M(\mathcal{A})$  introduced by Salvetti in [22]. By a combinatorial description we mean a construction, using the combinatorial data, of a regular CW-complex of dimension  $l$  which has the same homotopy type as that of  $M(\mathcal{A})$ . Note that this particular cell complex, which we denote by  $Sal(\mathcal{A})$ , is defined using the face poset and not the intersection lattice.

The  $k$ -cells of  $Sal(\mathcal{A})$  are in one-to-one correspondence with the pairs  $[F, C]$ , where  $F$  is a codimension  $k$  face of the given arrangement and  $C$  is a chamber whose closure contains  $F$ . A cell labeled  $[F_1, C_1]$  is contained in the boundary of another cell  $[F_2, C_2]$  if and only if  $F_1 \leq F_2$  in  $\mathcal{F}(\mathcal{A})$  and  $C_1, C_2$  are contained in the same chamber of (the arrangement)  $\mathcal{A}_{F_1}$  (with the attaching maps being homeomorphisms). The seminal result of Salvetti is:

**Theorem 1.5** (Salvetti [22]). *Let  $\mathcal{A}$  be an arrangement of real hyperplanes and  $M(\mathcal{A})$  be the complement of its complexification inside  $\mathbb{C}^l$ . Then there is an embedding of  $Sal(\mathcal{A})$  into  $M(\mathcal{A})$  moreover there is a natural map in the other direction which is a deformation retraction.*

Let  $E_1$  be the free  $\mathbb{Z}$ -module generated by the elements  $e_H$  for every  $H \in \mathcal{A}$ . Define  $E(\mathcal{A})$  to be the exterior algebra on  $E_1$ . For  $S = (H_1, \dots, H_p)$  ( $1 \leq p \leq n$ ), call  $S$  independent if  $rank(\cap S) := dim(H_1 \cap \dots \cap H_p) = p$  and dependent if  $rank(\cap S) < p$ . Notice the unfortunate clash of notations, this rank is different from the one used in the intersection lattice. Geometrically independence implies that the hyperplanes of  $S$  are in general position. Let  $I(\mathcal{A})$  denote the ideal of  $E$  generated by all  $\partial e_S := \partial(e_{H_1} \cdots e_{H_p})$ , where  $S$  is a dependent tuple and  $\partial$  is the differential in  $E$ .

**Definition 1.6.** The *Orlik-Solomon algebra* of a complexified central arrangement  $\mathcal{A}$  is the quotient algebra  $A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$ .

The following important theorem shows how cohomology of  $M(\mathcal{A})$  depends on the intersection lattice. It combines the work of Arnold, Brieskorn, Orlik and Solomon. For details and exact statements of their individual results see [18, Chapter 3, Section 5.4].

**Theorem 1.7.** *Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a complex arrangement in  $\mathbb{C}^l$ . For every hyperplane  $H_i \in \mathcal{A}$  choose a linear form  $l_i \in (\mathbb{C}^l)^*$ , such that  $\ker(l_i) = H_i$  ( $1 \leq i \leq n$ ). Then the integral cohomology algebra of the complement is generated by the classes*

$$\omega_i := \frac{1}{2\pi} \frac{dl_i}{l_i},$$

for  $1 \leq i \leq n$ . The map  $\gamma: A(\mathcal{A}) \rightarrow H^*(M(\mathcal{A}), \mathbb{Z})$  defined by

$$\gamma(e_H) \mapsto \omega_H$$

induces an isomorphism of graded  $\mathbb{Z}$ -algebras.

This theorem asserts that a presentation of the cohomology algebra of  $M(\mathcal{A})$  can be constructed from the data that are encoded by the intersection lattice.

**1.2. Oriented Matroids.** Let us first see how oriented matroids arise in the context of hyperplane arrangements. Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of hyperplanes in  $\mathbb{R}^l$  as before. Associated with every hyperplane  $H_i \in \mathcal{A}$ , there are two open half-spaces bounded by the hyperplane, which will be denoted by  $H_i^+$  (plus side) and  $H_i^-$  (minus side). Accordingly, we will use  $H_i^0$  to denote the hyperplane itself which can be called as the *zero* side. Using this we can subdivide the Euclidean space into strata of points that have the same position with respect to hyperplanes in  $\mathcal{A}$ . In order to achieve this we assign a sign vector  $X(v) = (X_1(v), \dots, X_n(v))$  to every point  $v \in \mathbb{R}^l$  as follows:

$$X_i(v) = \begin{cases} + & \text{if } v \in H_i^+ \\ 0 & \text{if } v \in H_i^0 \\ - & \text{if } v \in H_i^- \end{cases}$$

Let  $\mathcal{F}$  denote the set of all possible sign vectors that arise due to the induced stratification. It is not very difficult to verify that following properties are satisfied by  $\mathcal{F}$ . Obviously  $|\mathcal{F}| < 2^n$ . Since we are considering only central arrangements  $(0, \dots, 0) \in \mathcal{F}$ . It is also clear that  $-v$  realizes opposite sign configuration that of that of  $v$ . Hence, if  $X \in \mathcal{F}$  then  $-X \in \mathcal{F}$ . Suppose that a hyperplane  $H$  separates two points  $v$  and  $w$ , but the hyperplane  $H'$  does not. Then the line segment joining  $v$  and  $w$  intersects  $H$  in a point  $u$ . It also follows that  $X_H(u) = 0$ ,  $X_{H'} \neq 0$  and if there exists a hyperplane  $H''$  such that  $X_{H''} \neq 0$  then  $H''$  cannot contain both  $v$  and  $w$ . Finally, suppose that there are two points  $u$  and  $w$  with possibly different sign configurations and let  $L$  denote the line segment joining them. Then there exists  $z \in L$  such that if  $X_H(u) = 0$  then  $X_H(z) \neq 0$  and for all  $H$  such that  $X_H(w) \neq 0$  we have  $X_H(w) = X_H(z)$ .

The idea behind oriented matroids is to formalize the properties satisfied by the sign vectors of a hyperplane arrangement. Note that there are several other ways to define oriented matroids, these definitions depend only on the context in which oriented matroids arise. Essentially all the definitions are equivalent, for more details about the axioms defining oriented matroids and their equivalence see [2, Chapter 3].

Let  $E = \{1, \dots, n\}$  be the finite ground set for some  $n > 0$ . A *sign vector* is a function  $X: E \rightarrow \{+, 0, -\}$ , i.e., an assignment of signs to each element of  $E$ . The set of all possible possible sign vectors is denoted by  $\{+, 0, -\}^E$  and  $X_e$  stands for  $X(e)$  for all  $e \in E$ . The *support* of a vector  $X$  is  $\underline{X} = \{e \in E | X_e \neq 0\}$ ; its *zero set* is  $z(X) = E \setminus \underline{X}$ . The opposite of a vector  $X$  is  $-X$ , defined by  $(-X)_e = -(X_e)$ . The zero vector is  $\mathbf{0} = (0, \dots, 0)$ . The *composition* of two sign vectors  $X$  and  $Y$  is  $X \circ Y$  defined by

$$(X \circ Y)_e := \begin{cases} X_e & \text{if } X_e \neq 0 \\ Y_e & \text{otherwise} \end{cases}$$

The *separation set* of  $X$  and  $Y$  is  $S(X, Y) = \{e \in E | X_e = -Y_e \neq 0\}$ . With these terminologies in hand we can define oriented matroids using the *covector axioms*. These axioms generalize the geometric properties of the signed vectors (of a hyperplane arrangement) stated above.

**Definition 1.8.** A set  $\mathcal{L} \subset \{-, 0, +\}^E$  (of signed vectors) is the set of covectors of an oriented matroid if and only if it satisfies:

- (V0)  $\mathbf{0} \in \mathcal{L}$ ,
- (V1)  $X \in \mathcal{L} \Rightarrow -X \in \mathcal{L}$ ,
- (V2)  $X, Y \in \mathcal{L} \Rightarrow X \circ Y \in \mathcal{L}$ ,
- (V3) if  $X, Y \in \mathcal{L}$  and  $e \in S(X, Y)$  then there exists  $Z \in \mathcal{L}$  such that  $Z_e = 0$  and  $Z_f = (X \circ Y)_f = (Y \circ X)_f$  for all  $f \notin S(X, Y)$ .

There is a partial order on the sign vectors defined as follows:

$$Y \leq X \iff Y_e \in \{0, X_e\} \quad \forall e \in E$$

If  $\mathcal{L} \subset \{+, 0, -\}^E$  is a set of covectors of an oriented matroid then it inherits the above defined partial ordering to become a poset with the bottom element  $\mathbf{0}$ . The poset  $\hat{\mathcal{L}} := (\mathcal{L} \cup \{\hat{1}\}, \leq)$  is a lattice. The join in  $\hat{\mathcal{L}}$  of  $X$  and  $Y$  is  $X \circ Y = Y \circ X$  if  $S(X, Y) = \emptyset$ , and equals  $\hat{1}$  otherwise.

**Definition 1.9.** The lattice  $\mathcal{F}(\mathcal{L}) = (\hat{\mathcal{L}}, \leq)$  is called the *face lattice* of the oriented matroid  $\mathcal{L}$ . The maximal elements of  $\mathcal{L}$  are called *topes* (or *regions*). Let  $\mathcal{T}(\mathcal{L})$  denote the set of topes. The *rank* of  $\mathcal{L}$  is the length of a maximal chain in  $(\mathcal{L}, \leq)$ .

It is now clear that the faces of an arrangement are nothing but the sign vectors. They satisfy the above mentioned axioms for oriented matroids and the face poset is isomorphic to the oriented matroid with the sign ordering. Hence every hyperplane arrangement gives rise to an oriented matroid and such an oriented matroid is called as *realizable*.

*Remark 1.10.* If  $\mathcal{L}$  is a (linear) oriented matroid coming from a central hyperplane arrangement  $\mathcal{A}$  in  $\mathbb{R}^l$  then  $\mathcal{F}(\mathcal{L})$  is isomorphic to the face poset  $\mathcal{F}(\mathcal{A})$ . In particular the topes of  $\mathcal{L}$  correspond to the chambers of  $\mathcal{A}$ .

Given an oriented matroid  $(E, \mathcal{L})$  it can be shown that the set  $L = \{z(X) | X \in \mathcal{L}\}$  forms the collection of flats of the matroid underlying  $(E, \mathcal{L})$ . Recall that, for a matroid  $(E, L)$  on a finite ground set  $E = \{e_1, \dots, e_n\}$ , the dependent subsets of  $E$  that are minimal with respect to inclusion are called *circuits*. A *broken circuit* is an independent subset obtained by deleting the minimal element from a circuit. Finally, a *no-broken-circuit* set (or simply *nbc set*) is an independent subset that contains no broken circuits. The Orlik-Solomon algebra of a matroid is defined as the exterior algebra  $\mathbb{Z} < e_1, \dots, e_n >$  modulo the Orlik-Solomon ideal  $I(E)$ . This ideal is generated by the elements  $\sum_{i=1}^k (-1)^i e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_k$ , one for each circuit of  $L$ .

If  $(E, \mathcal{L})$  is realizable and  $\mathcal{A}$  is the corresponding hyperplane arrangement then  $L$  is the lattice of intersections of these hyperplanes and we have the following. A circuit of  $(\mathcal{A}, L)$  is a minimal set  $\mathcal{B}$  such that for every  $H \in \mathcal{B}$ , the set  $\mathcal{B} \setminus \{H\}$  is minimal with the property that the intersection of its hyperplanes equals  $\cap \mathcal{B}$ . In short, the circuits correspond to the dependent tuples of hyperplanes. We have already seen that the Orlik-Solomon algebra of this matroid is isomorphic to the cohomology ring of the complexified complement. A  $\mathbb{Z}$ -basis for this algebra is given by the nbc sets.

## 2. PSEUDOHYPERPLANE ARRANGEMENTS

It is not true in general that given an arbitrary oriented matroid there corresponds an arrangement of hyperplanes. Such oriented matroids are called as *non-realizable oriented*

*matroids.* The *Folkman-Lawrence topological representation theorem* [12] states that every oriented matroid is ‘almost’ realizable. Originally the topological representation theorem was stated in terms of *pseudo-hemisphere arrangements*. Later Arnaldo Mandel in his thesis [16] achieved much simplification using PL topology. He reproved the theorem in terms of *sphere systems* (popularly known as *arrangements of pseudospheres*). In this section we state the representation theorem and introduce a generalization of hyperplane arrangements.

We work with pseudohyperplanes, these are obtained by mildly deforming hyperplanes. We will show that their arrangements correspond to pseudosphere arrangements and hence to oriented matroids. Since the pseudohyperplanes can not be described by algebraic equations it is not possible to define their complexification. Moreover, there need not exist smooth structure on these deformed hyperplanes. Consequently even construction of a tangent bundle complement is not possible (see [10]). We address this question in this section. For each pseudohyperplane arrangement in  $\mathbb{R}^l$  we construct a subspace of  $\mathbb{R}^{2l}$  which generalizes the complexified complement. The main result of this section is the proof that this subspace deformation retracts on to the associated Salvetti complex.

**2.1. The topological representation theorem.** We first recall standard terminologies from PL topology. Let  $K$  and  $L$  denote two geometric simplicial complexes. A map (between the underlying spaces)  $f: ||K|| \rightarrow ||L||$  is said to be *piecewise linear (PL)* if it is linear with respect to some simplicial subdivision of  $K$ . A *PL homeomorphism* is a PL map which is also a homeomorphism of underlying spaces, a *PL embedding* is defined analogously. A *PL n-sphere* is a (geometric) simplicial complex which is PL homeomorphic to the boundary of a  $(n+1)$ -simplex, analogously a *PL n-ball* is PL homeomorphic to standard topological  $n$ -simplex. Following are some (relevant) well known facts in this field (we refer the reader to [21]). Recall that an embedding of a submanifold is locally flat if every point in the image has a neighborhood in which the submanifold is homeomorphic to a Euclidean subspace.

**Theorem 2.1.** *If  $f: M \rightarrow N$  is a PL embedding of the PL  $m$ -manifold  $M$  into the PL  $n$ -manifold  $N$  and  $m - n \neq 2$ , then  $f$  is locally flat.*

**Theorem 2.2.** *Let  $S^l$  denote the standard unit sphere in  $\mathbb{R}^{l+1}$ . If  $f: S^l \rightarrow \mathbb{R}^n$ ,  $n - l \neq 2$  is a locally flat embedding, then there exists a homeomorphism  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h \circ f$  is the inclusion map. The same conclusion holds for an embedding of  $\mathbb{R}^l$  into  $\mathbb{R}^n$ .*

A subset of the standard unit sphere is called a subsphere if it is homeomorphic to some lower dimensional sphere. We single out a class of subspheres that play an important role in defining more general types of arrangements.

**Lemma 2.3.** *For a  $(l-1)$ -subsphere  $S$  of  $S^l$  the following conditions are equivalent:*

- (1) *embedding of  $S$  is equivalent to the inclusion map,*
- (2) *embedding of  $S$  is equivalent to some PL  $(l-1)$ -subsphere of  $S^l$ ,*
- (3) *the closure of each connected component of  $S^l \setminus S$  is homeomorphic to the  $l$ -ball.*

The equivalence class of these subspheres is known as *tame*, all other embeddings are called *wild*. It is known that all embeddings of  $S^1$  into  $S^2$  are tame (the Schönflies theorem). However, there are wild 2-spheres in  $S^3$ , for example, the Alexander horned sphere.

**Definition 2.4.** A  $(l-1)$ -subsphere  $S$  in  $S^l$  satisfying any of the equivalent conditions in Lemma 2.3 is called a *pseudosphere* in  $S^l$ . The two connected components of  $S^l \setminus S$  are

its *sides*, denoted by  $S^+$  and  $S^-$ . The closures of the sides are called the *closed sides* (or *pseudohemispheres*)

We can now present the generalization of hyperplane arrangements that was used to prove the representation theorem.

**Definition 2.5.** A *signed arrangement of pseudospheres* in the standard unit sphere  $S^l \subseteq \mathbb{R}^{l+1}$  is a finite collection  $\mathcal{A} = \{(S_i^+, S_i^0, S_i^-) \mid i \in E\}$  where  $E = \{1, \dots, n\}$  such that

- (1) Each  $S_i^0$  is a pseudosphere in  $S^l$  with sides  $S_i^+$  and  $S_i^-$ .
- (2)  $S_I := \cap_{i \in I} S_i^0$  is a sphere, for all  $I \subseteq E$  ( $\emptyset$  is the  $(-1)$ -sphere).
- (3) If  $S_I \not\subseteq S_j$ , for some subset  $I$ , an index  $j$ , then  $S_I \cap S_j$  is a pseudosphere in  $S_I$  with sides  $S_I \cap S_j^+$  and  $S_I \cap S_j^-$ .

For the sake of notational simplicity we assume that both the sides of each pseudosphere are equipped with a sign and we will not explicitly mention it every time. Since each side has a sign attached to it one can define a sign function similar to that for hyperplane arrangements. Equivalently the position of each point  $x \in S^l$  with respect to each pseudosphere in the arrangement  $\mathcal{A}$  is given by a sign vector  $\sigma(x) \in \{+, 0, -\}^E$ , defined by

$$\sigma(x)_i = \begin{cases} + & \text{if } x \in S_i^+ \\ 0 & \text{if } x \in S_i^0 \\ - & \text{if } x \in S_i^- \end{cases}$$

The arrangement defines a stratification of the ambient sphere, and each strata is indexed by the sign vectors in  $\sigma(S^l)$ . One of the reasons why this type of generalization is necessary is the following:

**Theorem 2.6.** Let  $\mathcal{A}$  be a signed, essential arrangement of pseudospheres in  $S^l$ . Then

$$\mathcal{L}(\mathcal{A}) := \{\sigma(x) \mid x \in S^l\} \cup \{\mathbf{0}\} \subseteq \{+, 0, -\}^E$$

is the set of covectors of an oriented matroid and the rank of  $\mathcal{L}(\mathcal{A}) = l + 1$ .

Some of the topological properties of hyperplane arrangements also hold.

**Lemma 2.7** ([16] Lemma 3, page 201). Let  $\mathcal{A}$  be a signed and essential arrangement of pseudospheres in  $S^l$ . For every  $X \in \mathcal{L}(\mathcal{A}) \setminus \{\mathbf{0}\}$  the strata  $\sigma^{-1}(X)$  is an open cell of a regular cell decomposition  $\Delta(\mathcal{A})$  of  $S^l$ . The boundary of  $\overline{\sigma^{-1}(X)}$  is the union of all those  $\sigma^{-1}(Y)$  such that  $Y$  is properly covered by  $X$ . Furthermore, the mapping  $X \mapsto \{y \in S^l \mid \sigma(y) \leq X\}$  gives an isomorphism

$$\hat{\mathcal{L}}(\mathcal{A}) \cong \hat{\mathcal{F}}(\Delta(\mathcal{A}))$$

of the face lattice of  $\mathcal{L}(\mathcal{A})$  and face lattice of the regular cell complex  $\Delta(\mathcal{A})$ .

Two signed arrangements  $\mathcal{A} = \{S_1, \dots, S_n\}$  and  $\mathcal{A}' = \{S'_1, \dots, S'_n\}$  of pseudospheres in  $S^l$  are *topologically equivalent* ( $\mathcal{A} \sim \mathcal{A}'$ ) if there exists some homeomorphism  $h: S^l \rightarrow S^l$  such that  $h(S_i) = S'_i$  and  $h(S_i^+) = (S'_i)^+$  for all  $1 \leq i \leq n$ . This topological equivalence is combinatorially determined.

**Theorem 2.8.** Two signed arrangements  $\mathcal{A}$  and  $\mathcal{A}'$  in  $S^l$  are topologically equivalent if and only if  $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}')$ .

If the oriented matroid obtained from such an arrangement is realizable then we can retrieve hyperplane arrangements.

**Corollary 2.9.** *Let  $\mathcal{A} = \{S_1, \dots, S_n\}$  be a signed arrangement of pseudospheres in  $S^l$ . The oriented matroid  $\mathcal{L}(\mathcal{A})$  is realizable if and only if there exists a homeomorphism  $h: S^l \rightarrow S^l$  such that  $h(S_i) = S^l \cap H_i$ , where  $H_i$  is a codimension 1 subspace of  $\mathbb{R}^{l+1}$ , for every  $i$ .*

The topological representation theorem [12, Theorem 20], [16] and [2, Theorem 5.2.1], however, includes a converse to all of the above.

**Theorem 2.10 (Topological Representation Theorem).** *Let  $\mathcal{L} \subseteq \{+, 0, -\}^E$ . Then following conditions are equivalent:*

- (1)  $\mathcal{L}$  is the set of covectors of a (simple) oriented matroid of rank  $l$ .
- (2)  $\mathcal{L} = \mathcal{L}(\mathcal{A})$  for some signed arrangement  $\mathcal{A} = \{S_1, \dots, S_n\}$  of pseudospheres in  $S^{l-1}$ , which is essential and centrally symmetric and whose induced cell complex  $\Delta(\mathcal{A})$  is regular.

An arrangement is said to be *centrally symmetric* if each pseudosphere  $S \in \mathcal{A}$  is invariant under the antipodal mapping of  $S^l$  (and so are the sides, i.e.  $S_i^+ \mapsto S_i^-$  for every  $i$ ).

Let  $\mathcal{L}$  be the set of covectors of a rank  $l$  oriented matroid. According to Theorem 2.10 there corresponds a signed arrangement  $\mathcal{A} = \{S_1, \dots, S_n\}$  of pseudospheres in  $S^{l-1}$ , the unit sphere in  $\mathbb{R}^l$ . Since each pseudosphere  $S$  is centrally symmetric any pair of antipodal points  $x, -x \in S$  generates a line through the origin in  $\mathbb{R}^l$ . For  $S \in \mathcal{A}$  let  $H_S$  be the set of all rays from the origin passing through  $S$ . Specifically this set can be expressed as the cone over  $S$  as follows:

$$H_S = S \times [0, \infty) / \{S \times 0\}$$

The next result is now immediate and follows from Lemma 2.3.

**Lemma 2.11.** *Let  $S$  be a pseudosphere in the unit sphere  $S^{l-1}$  and  $H_S$  be the cone. Then there exists a homeomorphism of  $\mathbb{R}^l$  such that it maps  $H_S$  to a hyperplane passing through the origin.*

**Definition 2.12.** A *pseudohyperplane* in  $\mathbb{R}^l$  is defined as the cone over some pseudosphere in  $S^{l-1}$ . An *arrangement of pseudohyperplanes* is a finite collection  $\mathcal{A}$  of pseudohyperplanes in  $\mathbb{R}^l$  such that  $\{H \cap S^{l-1} \mid H \in \mathcal{A}\}$  is an arrangement of pseudospheres in  $S^{l-1}$ .

Given an arrangement  $\mathcal{A}$  of pseudospheres in  $S^{l-1}$  we denote by  $c\mathcal{A}$  the corresponding arrangement of pseudohyperplanes. A face of  $c\mathcal{A}$  is the cone over some face of  $\mathcal{A}$  and hence homeomorphic to an open polyhedral cone of 1 dimension higher.

**Example 2.13.** Consider the arrangement of circles in  $S^2$  as shown in Figure 1. It corresponds to the non-Pappus oriented matroid of rank 3. We first construct an arrangement of 8 circles in  $S^2$  such that points  $a, b, c$  are collinear and other three points  $a', b', c'$  are also collinear. According Pappus theorem the points  $d, e, f$  are also collinear. However we add the 9th circle which passes through the points  $d$  and  $f$  but not  $e$ . The resulting pseudosphere arrangement represents a non-oriented matroid. The corresponding pseudo-plane arrangement in  $\mathbb{R}^3$  is obtained by letting rays from the origin pass through each of these 9 circles.

We state the following corollary for the sake of completeness.

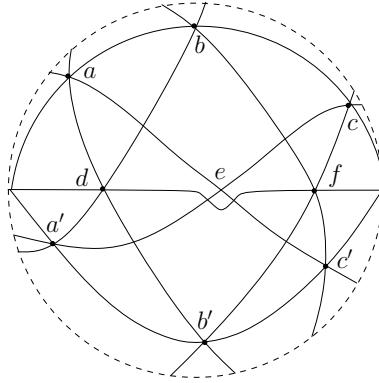


FIGURE 1. Non-Pappus arrangement

**Corollary 2.14.** Let  $\mathcal{L} \subseteq \{+, 0, -\}^E$  be the set of covectors of a oriented matroid of rank  $l$ . Then there exists an arrangement of pseudohyperplanes  $c\mathcal{A}$  such that

$$\mathcal{F}(c\mathcal{A}) \cong (\mathcal{L}, \leq).$$

**Remark 2.15.** In the literature related to topological representation theorem the word *pseudohyperplane* is used for the (codimension 1) projective space obtained by applying the antipodal map. However here we have used this term for a tame embedding of a hyperplane. Miller has also used this word for topologically deformed hyperplanes in [17] where he describes a slightly different topological representation for a certain class of oriented matroids.

**2.2. The connected complement.** Throughout this section we fix an arbitrary simple oriented matroid  $\mathcal{L}$  of rank  $l - 1$ , let  $\mathcal{A}$  and  $c\mathcal{A}$  denote the corresponding arrangements of pseudospheres (in  $S^{l-1}$ ) and pseudohyperplanes (in  $\mathbb{R}^l$ ) respectively. Our aim is to construct a connected subspace of  $\mathbb{R}^{2l}$  and then show that it has the homotopy type of a simplicial complex that is determined by the oriented matroid.

Let  $c\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of pseudohyperplanes in  $\mathbb{R}^l$ . For every  $x \in \mathbb{R}^l$  the *arrangement restricted* at  $x$  is

$$c\mathcal{A}_x := \{H \in c\mathcal{A} \mid x \in H\}.$$

Define the *local complement* at  $x$  as:

$$M(c\mathcal{A}_x) := \mathbb{R}^l \setminus c\mathcal{A}_x.$$

Finally, define the *complexified complement* of  $c\mathcal{A}$  as:

$$M(c\mathcal{A}) := \coprod_{x \in \mathbb{R}^l} \mathbb{R}^l \setminus c\mathcal{A}_x = \{(x, v) \mid x \in \mathbb{R}^l, v \in M(c\mathcal{A}_x)\} \subseteq \mathbb{R}^{2l}$$

**Lemma 2.16.** The space  $M(c\mathcal{A})$  is connected.

*Proof.* For any two points  $(x_1, v_1)$  and  $(x_2, v_2)$  we show that there is path in  $M(c\mathcal{A})$  connecting these two points. Let  $\{\alpha(t) \mid t \in [0, 1]\}$  be a continuous path starting from  $x_1$  and ending at  $x_2$  in  $\mathbb{R}^l$ . Let  $F$  be the face containing  $x_1$ . The local complement  $M(c\mathcal{A}_{x_1})$  is disconnected and its components correspond to chambers of the arrangement  $c\mathcal{A}_x$ . Let  $C$  be the chamber of  $c\mathcal{A}$  containing  $v_1$  and  $C_{x_1}$  be the chamber of  $c\mathcal{A}_{x_1}$  containing  $C$ . Now for every  $y \in F$

the local complement  $M(c\mathcal{A}_y)$  contains connected component  $C_y$  such that  $C \subseteq C_y$ . Therefore  $v_1 \in C_y$  for every  $y \in F$  and  $\{(\alpha(t) \cap F, v_1) \mid t \in [0, 1]\}$  is a continuous path in  $M(c\mathcal{A})$ , call it  $\beta_F$ . Now we have two cases to deal with.

**Case 1:** Let  $G$  be a face such that  $F$  covers  $G$  and  $Im(\alpha) \cap G \neq \emptyset$ . As  $G \leq C$  in the face poset we have that for every  $y \in G$  there is a connected component  $C_y$  of  $M(c\mathcal{A}_y)$  that contains  $C$ . Hence  $\{(\alpha(t) \cap G, v_1) \mid t \in [0, 1]\}$  is also a continuous path, denote it by  $\beta_G$ .

**Case 2:** Let  $G$  be a face such that  $F$  is covered by  $G$  and  $Im(\alpha) \cap G \neq \emptyset$ . Hence for every  $y \in G$  the local complement  $M(c\mathcal{A}_y)$  has a component  $C_y$  that contains  $C$  and  $G \circ C$ . Let  $z \in G \circ C$  and  $\gamma_G$  be a continuous path in  $C_y$  joining  $z$  and  $v_1$ . Let  $\beta_G$  denote the path which is made up of concatenating  $\gamma_G$  with  $\{(\alpha(t) \cap G, z) \mid t \in [0, 1]\}$  (appropriately).

Continuing this process one can construct a path  $\beta$ , by concatenating the paths  $\beta_G$  (for every face  $G$  that intersects with the path  $\alpha$ ) which joins any two points. Hence  $M(c\mathcal{A})$  is path connected.  $\square$

We now want to construct an open covering of the space  $M(c\mathcal{A})$ . First we state some more terminology and results from topological embeddings that we need. A connected codimension 1 submanifold  $N$  of a manifold  $M$  is *two-sided* if there is a connected open neighborhood of  $U$  of  $N$  in  $M$  such that  $U \setminus N$  has exactly two components each of which is open in  $M$ . Further,  $N$  is said to be *bicollared* in  $M$  if it has an open neighborhood homeomorphic to  $N \times (-1, 1)$  with  $N$  itself corresponding to  $N \times \{0\}$ . We will use the following theorem originally due to M. Brown in 1964.

**Theorem 2.17.** *Let  $N$  be a locally flat, connected, two-sided, codimension 1 submanifold of  $M$ . Then  $N$  is bicollared in  $M$ .*

For every  $F \in \mathcal{F}(c\mathcal{A}) \setminus \{\mathbf{0}\}$  let  $\tilde{F}$  be the face of  $\mathcal{A}$  such that  $\tilde{F} = F \cap S^{l-1}$ . If  $\sigma$  is the function assigning signs to every face then

$$\begin{aligned}\sigma(F) &= \sigma(\tilde{F}) & \forall F \in \mathcal{F}(c\mathcal{A}) \setminus \{\mathbf{0}\} \\ \sigma(\mathbf{0}) &= (0, \dots, 0)\end{aligned}$$

As stated before, such a face  $F$  is just the cone over  $\tilde{F}$ , hence homeomorphic to an open polyhedral cone in  $\mathbb{R}^l$ . For a tope  $T$  let  $V_T$  denote the corresponding chamber in  $\mathbb{R}^l$ . For the sign vector  $\mathbf{0}$  let  $V_{\mathbf{0}}$  be the open unit ball. Let  $F$  be a face which is neither a chamber nor  $\mathbf{0}$ . Let  $H_F$  be the support of  $F$  and  $B(H_F)$  be its bicollar (pseudohyperplanes satisfy the hypothesis of Theorem 2.17). Let  $V_F$  be the portion of  $B(H_F)$  that contains  $F$  and intersects only those faces whose closures contain  $F$ .

From the above construction it is easy to prove the following lemma which explains properties of these open sets.

**Lemma 2.18.** *With the notation as above, the following statements are true:*

- (1) *For every  $F \in \mathcal{F}(c\mathcal{A})$  the open set  $V_F$  contains  $F$  and is homeomorphic to  $\mathbb{R}^l$ .*
- (2) *If  $F \leq F'$  in  $\mathcal{F}(\mathcal{A})$  then  $V_F \cap V_{F'} \neq \emptyset$  and  $F \not\subseteq V_{F'}$ .*
- (3) *If  $F$  and  $F'$  are not comparable in  $\mathcal{F}(c\mathcal{A})$  then  $V_F \cap V_{F'} = \emptyset$ .*

For every pair  $(F, T)$ , where  $F$  is a face and  $T$  is a tope covering it, define a subset of  $M(c\mathcal{A})$  as follows:

$$W(F, T) := V_F \times V_T$$

**Theorem 2.19.** *The collection  $\{W(F, T) \mid (F, T) \in \mathcal{F} \times \mathcal{T}, F \leq T\}$  forms an open covering of  $M(c\mathcal{A})$  and whenever these open sets intersect the intersection is contractible.*

*Proof.* Let  $(x, v) \in M(c\mathcal{A})$  be any point. Therefore there is some face  $F$  such that  $x \in F \subseteq V_F$  and some chamber  $C$  such that  $v \in C \subset M(c\mathcal{A}_x)$ . These sets are open and contractible because they are products of open and contractible subsets. The intersections are contractible for the same reasons.  $\square$

Since the hypothesis of the Nerve Lemma [15, Theorem 15.1] is satisfied, the nerve of this open covering has the homotopy type of  $M(c\mathcal{A})$ . We can also deduce the criterion for their intersections to be non-empty as it is needed to identify the simplices.

**Lemma 2.20.**

$$W(F_1, T_1) \cap W(F_2, T_2) \neq \emptyset \iff F_1 \leq F_2 \text{ and } T_2 = F_2 \circ T_1$$

*Proof.* By construction of these open sets we have,

$$W(F_1, T_1) \cap W(F_2, T_2) = (V_{F_1} \cap V_{F_2}) \times (V_{T_1} \cap V_{T_2})$$

Clearly  $V_{F_1} \cap V_{F_2} \neq \emptyset$  if and only if  $F_1 \leq F_2$ . We also need the other intersection to be nonempty,

$$\begin{aligned} V_{T_1} \cap V_{T_2} \neq \emptyset &\iff T_1 \cap T_2 \neq \emptyset \\ &\iff T_2 = F_2 \circ T_1 \text{ or } T_2 = T_1. \quad \square \end{aligned}$$

Let us first construct the nerve as an abstract simplicial complex.

**Definition 2.21.** Let  $\mathcal{L}$  be the set of covectors of an oriented matroid and let  $\mathcal{T}$  be the set of all topes. Define a partial order on the set of all pairs  $(X, T)$  for which  $X \in \mathcal{L}$ , and  $T \in \mathcal{T}$ , by the following rule:

$$(X_2, T_2) \leq_S (X_1, T_1) \iff X_1 \leq X_2 \text{ and } X_2 \circ T_1 = T_2$$

The **Salvetti complex**  $Sal(\mathcal{L})$  is the regular cell complex having this poset as its face poset.

**Theorem 2.22.** *Let  $\mathcal{L}$  denote the set of covectors of an oriented matroid and  $c\mathcal{A}$  be the associated arrangement of pseudohyperplanes. If  $M(c\mathcal{A})$  is the associated space then*

$$M(c\mathcal{A}) \simeq Sal(\mathcal{L}).$$

### 3. METRICAL HEMISPHERE COMPLEXES

Recall that a central arrangement of hyperplanes decomposes the ambient Euclidean space into open polyhedral cones. As a matter of fact every hyperplane arrangement is a normal fan of a very special polytope known as the zonotope. Zonotopes can be defined in various ways: for example, projections of cubes, Minkowski sums of line segments, dual (polar) of hyperplane arrangements etc. For more on the relationship between zonotopes and hyperplane arrangements see [25, Lecture 7].

**Definition 3.1.** A *zonotope* is a polytope all of whose faces are centrally symmetric (equivalently every 2-face is centrally symmetric). A *zonotopal cell* is a (closed)  $k$ -cell such that its face poset is isomorphic to the face poset of a  $k$ -zonotope for some  $k$ .

The face poset of a zonotope has some special combinatorial properties, the most important of which is the product structure. This product is basically the one on the face poset of a hyperplane arrangement or on the set of covectors of an oriented matroid. Following result clarifies the relationship between these three structures (see [25, Corollary 7.17]).

**Theorem 3.2.** *There is a natural bijection between the following three families:*

- (1) *the faces of a (central and essential) hyperplane arrangement in  $\mathbb{R}^l$ ,*
- (2) *the non-empty faces of zonotope in  $\mathbb{R}^l$  (which arises as a dual of a hyperplane arrangement).*
- (3) *signed covectors of a simple (realizable) oriented matroid of rank  $l$ .*

We now extend the above correspondence to non-realizable oriented matroids. We have already shown that these oriented matroids correspond to pseudo arrangements. We now generalize zonotopes. In order to do this we use the language of *metrical-hemisphere complexes*. These cell complexes possess all the essential combinatorial properties of a zonotope. The metrical-hemisphere complexes (MH-complexes for short) were first introduced in [23] where Salvetti generalized his construction and proved an analogue of Deligne's theorem for oriented matroids.

Let  $Q$  denote a connected, regular, CW complex (and  $|Q|$  be the underlying space). The 1-skeleton of such a complex  $Q$  is a graph  $G(Q)$  with no loops (abbreviated to  $G$  if the context is clear). The vertex set of this graph will be denoted by  $VG$  and the edge set by  $EG$ . An edge-path in  $G(Q)$  is a sequence  $\alpha = (l_1, \dots, l_n)$  of edges that correspond to a connected path in  $|Q|$ . The inverse of a path is again a path  $\alpha^{-1} = (l_n, \dots, l_1)$ . Two paths are composed by concatenation if ending vertex of one of the paths is the starting vertex of another. The distance  $d(v, v')$  between two vertices will be the least of the lengths of paths joining  $v$  to  $v'$ . Given an  $i$ -cell  $e^i \in Q$ ,  $Q(e^i) := \{e^j \in Q : |e^j| \subset |e^i|\}$  and let  $V(e^i) = VG \cap Q(e^i)$ .

**Definition 3.3.** A regular CW complex  $Q$  is a QMH-complex (quasi-metrical-hemisphere complex) if and only if there exist two maps

$$\underline{\omega}, \bar{\omega} : VG \times Q \rightarrow VG$$

such that for all  $v \in VG, e^i \in Q$  following properties are satisfied.

- (1)  $\underline{\omega}(v, e^i) \in V(e^i)$  and  $d(v, \underline{\omega}(v, e^i)) = \min\{d(v, u) | u \in V(e^i)\}$ .
- (2)  $\bar{\omega}(v, e^i) \in V(e^i)$  and  $d(v, \bar{\omega}(v, e^i)) = \max\{d(v, u) | u \in V(e^i)\}$ .
- (3)  $d(v, \bar{\omega}(v, e^i)) = d(v, u) + d(u, \bar{\omega}(v, e^i))$  for all  $u \in V(e^i)$ .

This definition imposes a strong restriction on the 1-skeletons of such complexes (see [23, Proposition 1]).

**Lemma 3.4.** *If  $Q$  is a QMH-complex then each circuit in  $G$  has an even number of edges.*

The next corollary follows from the above lemma and the definition of a zonotope.

**Corollary 3.5.** *Let  $Q$  be a closed  $k$ -cell which is a QMH-complex. Then  $Q$  is a zonotopal cell.*

For any  $e^i \in Q$ , indicate by  $G(e^i) \subset G(Q)$  the subgraph corresponding to the 1-skeleton of  $e^i$  and by  $d_{G(e^i)}$  the distance computed using  $G(e^i)$ .

**Definition 3.6.** A regular CW complex will be called a LMH-complex (local-metrical-hemisphere complex) if and only if each  $Q(e^i)$  is a QMH-complex with respect to  $d_{G(e^i)}$ . Moreover, the following compatibility condition also holds: if  $e^k \in Q(e^i) \cap Q(e^j), v \in V(e^i) \cap V(e^j)$  then

$$\underline{\omega}_{(e^j)}(v, e^k) = \underline{\omega}_{(e^i)}(v, e^k), \quad \overline{\omega}_{(e^j)}(v, e^k) = \overline{\omega}_{(e^i)}(v, e^k).$$

Here,  $\underline{\omega}_{(e^j)}, \overline{\omega}_{(e^j)}$  are defined similar to  $\underline{\omega}, \overline{\omega}$  but using  $d_{G(e^j)}$ .

Finally,  $Q$  will be called a **MH-complex** if  $Q$  is both a QMH-complex and a LMH-complex and for all  $e^i \in Q, e^j \in Q(e^i), v \in V(e^i)$

$$\underline{\omega}(v, e^j) = \underline{\omega}_{(e^i)}(v, e^j), \quad \overline{\omega}(v, e^j) = \overline{\omega}_{(e^i)}(v, e^j).$$

*Remark 3.7.* Note that the 1-skeleton of a MH-complex has very special properties with respect to the distance. It is not enough to have a cell complex all of whose cells are zonotopal. Here are two examples that illustrate the special nature of MH-complexes.

Consider a 2-dimensional cell complex made up of two 1-cells and two 2-cells. The 1-cells are attached to an octagonal 2-cell. There is a one more trapezoidal 2-cell whose three 1-cells are attached to three 1-cells in the boundary of the octagonal cell as shown in Figure 2. The resulting complex is QMH but not LMH. Consider the 1-cell labeled by  $e$  in the figure. There are two vertices, namely  $v_1, v_2$ , in its boundary. Considering  $e$  as a member of the trapezoidal cell we see that the vertex  $v_1$  is closest to the vertex  $v_4$ . On the other hand as a member of the octagonal 2-cell vertex  $v_2$  is closest to  $v_4$ .

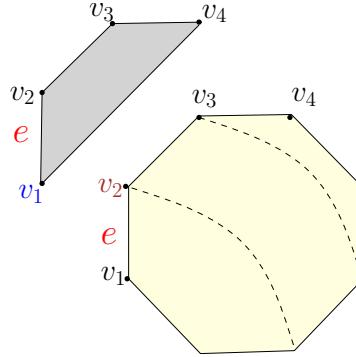


FIGURE 2. A QMH complex without LMH structure.

The next example shows a cell complex (Figure 3) obtained by removing the trapezoidal 2-cell from the first example. The resulting cell complex is both QMH and LMH but not a MH-complex. Consider the 1-cell labeled by  $e$ , there are two boundary vertices  $v_1, v_2$ . Considering  $e$  as a member of the octagonal cell the vertex  $v_2$  is closest to  $v_3$ . But in the whole complex the boundary vertex of  $e$  closest to  $v_3$  is  $v_1$ .

The following lemma establishes the combinatorial connection between zonotopes and MH-complexes. It states that the distance between any two vertices is the same no matter how it is measured, either locally or globally (see [23, Proposition 5]).

**Lemma 3.8.** Let  $Q$  be a MH-complex,  $e^i \in Q, v, v' \in V(e^i)$ . Then  $d(v, v') = d_{G(e^i)}(v, v')$ .

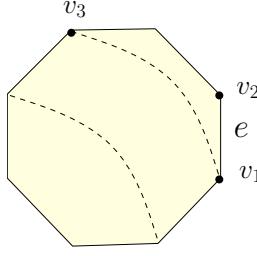


FIGURE 3. A QMH and LMH-complex which is not a MH-complex

*Proof.* Let  $\alpha = (l_1, \dots, l_n)$  be a minimal path of  $G(e^i)$  between  $v$  and  $v'$  (so  $d_{G(e^i)}(v, v') = n$ ). Let  $v_{j-1}, v_j$  be the vertices of  $l_j$  ordered according to the orientation of  $\alpha$  from  $v$  to  $v'$ . Since  $\alpha$  is minimal in  $G(e^i)$  and  $Q$  is a MH-complex we have,

$$\underline{\omega}_{(e^i)}(v, l_j) = v_{j-1} = \underline{\omega}(v, l_j)$$

Hence,  $d_{G(e^i)}(v, v_j) = d_{G(e^i)}(v, v_{j-1}) + 1$  and  $d(v, v_j) = d(v, v_{j-1}) + 1$  for  $j = 1, \dots, n$ .  $\square$

Given a pseudohyperplane arrangement  $c\mathcal{A}$  in  $\mathbb{R}^l$  (corresponding to an oriented matroid) and  $\mathcal{F}(c\mathcal{A})$  as its face poset let  $\mathcal{F}^*$  denote the dual face poset. By  $(\mathbb{R}^l, \mathcal{F}^*)$  we denote the cell complex (embedded in  $\mathbb{R}^l$ ) which is dual to the induced stratification. For every chamber of  $c\mathcal{A}$  there corresponds a 0-cell in  $(\mathbb{R}^l, \mathcal{F}^*)$  (in general, for a  $k$ -face  $F$  there is a  $l - k$ -cell  $F^*$ ). We now state the theorem that generalizes the relationship between hyperplane arrangements and zonotopes.

**Lemma 3.9.** *Let  $c\mathcal{A}$  be a pseudohyperplane arrangement corresponding to an oriented matroid, let  $\mathcal{F}(c\mathcal{A})$  be the face poset and  $\mathcal{F}^*$  be its dual then the cell complex  $(\mathbb{R}^l, \mathcal{F}^*)$  is a MH-complex.*

*Proof.* Observe that the distance between two dual vertices is equal to the number of pseudohyperplanes that separate corresponding chambers. The action of faces on chambers is given by composition of corresponding covectors. The proof that  $(\mathbb{R}^l, \mathcal{F}^*)$  is a MH-complex follows from verification of the axioms, it is given in [10, Theorem 3.3.14].  $\square$

We should note here that the MH-complex structures on the closed unit ball in  $\mathbb{R}^l$  are in one-to-one correspondence with simple oriented matroids of rank  $l$ . This can be done by proving that the cells of this MH-complex satisfy covector axioms. The proof is technical and will appear elsewhere.

For the sake of completeness we will explicitly describe the cells of the Salvetti complex. The 0-cells correspond to topes of the oriented matroid which we denote by  $[T, T]$ . Let  $X$  be a covector which corresponds to a  $(l - k)$ -face  $F_X$  of the corresponding pesudohyperplane arrangement. For every such covector  $X$  and every tope  $T$  such that  $X \leq T$  there corresponds a  $k$ -cell  $[X, T]$  which is homeomorphic to  $F_X^*$ . The boundary of such a cell is given by:

$$\partial[X, T] = \bigcup_{X < Y} [Y, Y \circ T].$$

The Salvetti complex has an oriented 1-skeleton by directing an edge  $[X, T]$  from  $[T, T]$  to  $[T', T']$ . Where  $T$  and  $T'$  are topes greater than  $X$ .

The oriented 1-skeleton of a Salvetti complex has the following additional combinatorial structure:

**Lemma 3.10.** *Let  $\text{Sal}(\mathcal{L})_1$  denote the oriented 1-skeleton of a Salvetti complex associated to an oriented matroid  $\mathcal{L}$ . Define a map  $\phi$  on the closed 1-cells by sending  $[X, T] \mapsto [X, T']$  where  $T, T'$  are topes covering  $X$ . If  $\mathcal{C}$  is the set of all circuits of the graph underlying  $\text{Sal}(\mathcal{L})_1$  then the triple  $(\text{Sal}(\mathcal{L})_1, \mathcal{C}, \phi(\mathcal{C}))$  satisfies the circuit-cocircuit axioms (dual pairs axioms) of an oriented matroid.*

#### 4. TOPOLOGY OF THE CONNECTED COMPLEMENT

In this section we further investigate topology of the space  $M(c\mathcal{A})$ . The results in this section are not new. Most of these results are already proved for hyperplane arrangements. We state them in the general context of oriented matroids.

**4.1. Salvetti complex.** We start by elaborating on the relationship between an oriented matroid and the combinatorics of the cells of the associated Salvetti complex.

**Theorem 4.1.** *Let  $\mathcal{L}$  be an oriented matroid and  $\text{Sal}(\mathcal{L})$  be the associated Salvetti complex then we have the following:*

- (1) *The complex  $\text{Sal}(\mathcal{L})$  is a MH-complex.*
- (2) *The number of 0-cells of  $\text{Sal}(\mathcal{L})$  is equal to the number of its  $l$ -cells which is also equal to the number of topes of  $\mathcal{L}$ .*
- (3) *Every chain in the barycentric subdivision of  $\text{Sal}(\mathcal{L})$  corresponds to pair consisting of a chain in  $(\mathcal{L}, \leq)$  and a tope.*
- (4) *The geometric realization of  $(\mathcal{L}, \leq)$  is a retract of  $\text{Sal}(\mathcal{L})$ .*
- (5)  $\chi(\text{Sal}(\mathcal{L})) = 0$ .
- (6) *The homeomorphism type of  $M(c\mathcal{A})$  is completely determined by  $\mathcal{L}$ .*

*Proof.* The first statement follows from the construction of the Salvetti complex. The 1-skeleton of this complex is obtained by doubling the edges in the 1-skeleton of  $(\mathbb{R}^l, \mathcal{F}^*)$ . We refer the reader to Deshpande [10, Theorem 3.3.15] and Salvetti [23, Proposition 10] for a detailed proof. Second statement follows from the Definition 2.21, since the 0-cells correspond chamber-chamber (tope-tope) pairs. The top dimensional cells correspond to pairs of the form  $[\mathbf{0}, T]$  for every tope  $T$ . The proof of the third statement is also clear from the definition. As for statement (4) the map  $X \mapsto [X, X \circ T]$  is an inclusion of  $\Delta(\mathcal{F}(\mathcal{L}))$  into  $\text{Sal}(\mathcal{L})$  and the map in the other direction defined by  $[X, T] \mapsto X$  is a retraction. For (5) see [10, Theorem 3.3.6]. The proof of the last statement is same as that of hyperplane arrangements given by Björner-Ziegler [3, Theorem 5.3].  $\square$

A path in the (regular) cell complex is a sequence of consecutive edges and its length is the number of edges. A minimal path is path of shortest length among all the paths that join its end points. In case of a an oriented 1-skeleton by a positive path we mean a path all of whose edges have same direction.

**Lemma 4.2.** *With the notation as before any two positive minimal paths in the 1-skeleton of  $\text{Sal}(\mathcal{L})$  that have same initial as well as terminal vertex are homotopic relative to  $\{0, 1\}$ .*

*Proof.* Given two positive minimal paths  $\alpha, \beta$  in  $\text{Sal}(\mathcal{L})$  with the same end points apply the retraction map to get paths in  $\Delta(\mathcal{F}^*)$ . Observe that no two edges of these two paths are sent to a same edge of  $\Delta(\mathcal{F}^*)$ . The conclusion follows from the fact that  $\Delta(\mathcal{F}^*)$  is contractible. See also Deshpande [10, Theorem 3.5.5], Salvetti [23, Theorem 17] for a proof. The proof by Cordovil [6, Theorem 2.4] is completely combinatorial.  $\square$

Given an oriented matroid  $\mathcal{L}$  let  $\mathcal{G}^+$  denote the associated *positive category*. It is defined to be the category of directed paths in  $\text{Sal}(\mathcal{L})$ . The objects of this category are the topes (vertices of the Salvetti complex) and morphisms are directed homotopy classes of positive paths (i.e. two such paths are connected by a sequence of substitutions of minimal positive paths). Let  $\mathcal{G}$  denote the arrangement groupoid. It is basically the fundamental groupoid of the associated Salvetti complex (or the category of fractions of  $\mathcal{G}^+$ ).

**Theorem 4.3.** *The associated canonical functor  $J: \mathcal{G}^+ \rightarrow \mathcal{G}$  is faithful on the class of minimal positive paths.*

*Proof.* We have already seen that any two minimal positive paths with same end points represent the same morphism in  $\mathcal{G}$ . We now have to show that they also represent the same morphism in  $\mathcal{G}^+$ .

For a chamber  $C$  let  $C^\# = \{\mathbf{0}\} * C$ , the chamber opposite to  $C$ . Let  $\alpha, \beta$  be two minimal positive paths from  $C$  to another chamber  $D$ . Since  $[C, C]$  and  $[D, D]$  are vertices of the  $l$ -cell  $[\mathbf{0}, C]$  the paths  $\alpha$  and  $\beta$  are contained in its boundary. By the definition of MH-complex these two paths can be extended to two minimal positive paths from  $[C, C]$  to  $[C^\#, C^\#]$  which are clearly equivalent in  $\mathcal{G}^+$ .  $\square$

Delucchi in his thesis [8] introduced the theory of Salvetti-type diagram models also in order to characterize and classify covering spaces of the complexified complement of a hyperplane arrangement. This homotopy theoretic technique not only works in case of non-realizable oriented matroids but generalized even to submanifold arrangements. The covering spaces can be realized as the homotopy colimit of certain diagrams of spaces defined using covering groupoids of  $\mathcal{G}$ . Stating these results requires some terminology from homotopy theory and would be a digression hence we refer the reader to [10, Chapters 3, 5] for precise statements.

**4.2. Cohomology of the complement.** The cohomology algebra of the complexified complement of a hyperplane arrangement is determined by its intersection lattice. We have seen in Section 1.2 that the construction of the OS algebra is completely combinatorial and works for any matroid. Hence even in case of a non-realizable oriented matroid its underlying matroid has the associated OS-algebra. It was shown by Gel'fand and Rybnikov [13, Theorem 5] that the cellular cohomology ring of a Salvetti complex is isomorphic to the associated OS-algebra. Their technique was generalized by Björner and Ziegler [3, Corollary 7.3] to arbitrary complex arrangements to give a completely combinatorial proof of the Brieskorn-Orlik-Solomon theorem.

It was proved by Dimca and Papadima [11] that the complement of a hyperplane arrangement is minimal, i.e., it has the homotopy type of a CW complex such that number of  $k$ -cells is equal to  $k$ -th Betti number. From the work of Delucchi [9] it follows that even in case of a non-realizable oriented matroid the associated Salvetti complex is minimal. This is done by constructing maximum acyclic matchings of the Salvetti complex such that its critical cells

are in one-to-one correspondence with the topes, this correspondence is achieved via the nbc sets [9, Proposition 2, Lemma 5.10].

**4.3. Simplicial oriented matroids.** We now turn to simplicial arrangements, that is, arrangements in which every chamber is a cone over an open simplex. Alternately, an oriented matroid is simplicial if  $\mathcal{L} \setminus \mathbf{0}$  is isomorphic to the face poset of a simplicial decomposition of the sphere (or for everytope  $T$  the interval  $[\mathbf{0}, T]$  is Boolean).

Recall that there is an ordering on the topes of an oriented matroid induced by the corresponding chambers. Fix a tope  $T$  for any other tope  $S$  define the distance  $d(S, T)$  between  $S$  and  $T$  to be the number of pseudohyperplanes that separate the corresponding chambers. Now, for another tope  $S'$ ,  $S' \prec_T S$  if and only if  $d(S', T) \leq d(S, T)$  is a partial order. Denote this poset by  $\mathcal{P}_T(\mathcal{L})$ .

**Theorem 4.4.** *Let  $\mathcal{L}$  be an oriented matroid and  $c\mathcal{A}$  be the associated pseudohyperplane arrangement. Then the following are equivalent*

- (1)  $\mathcal{L}$  is simplicial.
- (2) The positive category admits the Deligne normal form.
- (3) The tope poset  $\mathcal{P}_T(\mathcal{L})$  is a lattice for every tope  $T$ .

All of the above conditions imply that the space  $M(c\mathcal{A})$  is  $K(\pi, 1)$ .

*Proof.* The Deligne normal form is a particular factorization of loops in the Salvetti complex into compositions of positive paths.  $1 \Rightarrow 2$  is originally due to Deligne [7] (and a reproof by Paris [19]); both these proofs are for realizable oriented matroids. For non-realizable oriented matroids the proof was given by Cordovil [5, Theorem 4.1] and by Salvetti [23, Theorem 33].  $2 \Rightarrow 1$  is due to Paris [20]. For  $1 \iff 3$  see [1] and the proof of  $3 \iff 1$  is in Delucchi's thesis [8, Theorem 6.4.6, Lemma 6.5.2].

In fact, in this case, using Charney's arguments [4] it can be shown that the fundamental group of the Salvetti complex is an example of a Garside group.  $\square$

Obvious examples of arrangements that were not covered by Deligne's theorem are the simplicial arrangements of pseudolines. A simplicial arrangement of pseudolines in  $\mathbb{R}P^2$  consists of a finite family of simple closed curves such that every two curves have precisely one point in common and every 2-face is isomorphic to a triangle. By applying the coning process we get an arrangement of (non-stretchable) pseudoplanes in  $\mathbb{R}^3$  whose face poset correspond to a rank 3 non-realizable oriented matroid. The Salvetti complex associated to such oriented matroids is a  $K(\pi, 1)$  space. In fact there are at least seven infinite families of non-stretchable simplicial arrangement of pseudolines are known, see [14, Chapter 3] for details and examples of such arrangements.

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